Int. J. Heat Mass Transfer. Vol. 16, pp. 1816–1819. Pergamon Press 1973. Printed in Great Britain

EXACT SOLUTION BY PERTURBATION METHOD FOR PLANAR SOLIDIFICATION OF A SATURATED LIQUID WITH CONVECTION AT THE WALL

R. I. PEDROSO*
Philips Laboratories, Briarcliff Manor, New York, U.S.A.

and

G. A. DOMOTO
Department of Mechanical Engineering, Columbia University, New York, New York, U.S.A.

(*Received* 30 *November* 1972 *and in revised form* 22 *March* 1973)

INTRODUCTION

IN A PREVIOUS paper [1] a parameter perturbation technique was introduced for spherical solidification of a saturated liquid with the wall temperature fixed. In that work, only the first three terms of the solution were presented. The amount of algebraic work made it difficult to calculate more terms in the solution. The present investigation introduces a technique to calculate as many terms as desired in this type of perturbation solution. The case of planar solidification of a saturated liquid with convection at the wall is considered. The properties of the solidified material and the freezing temperature are assumed constant. Extensions should be possible for other types of boundary conditions and for outward and partial inward spherical as well as cylindrical solidification. The perturbation parameter used is a qualitative measure of the sensible heat in the solidified material relative to the latent heat of fusion liberated at the freezing front. It is shown that exact solutions can be obtained for values of this parameter less than or equal to one.

The technique presented in this paper does not eliminate the amount of algebraic work encountered when many terms of the perturbation solution are calculated. Instead, the difficulties are rearranged in such a manner that a digital computer can be used to advantage. No finite difference or any other numerical methods are used to calculate the coefficients of integer powers of the perturbation parameter for the temperature distribution, the freezing-front speed and its inverted series (i.e. the derivative of the time with respect to the freezing-front position). Numerical integration must then be used to obtain the time as a function of the freezing-front position.

* Part of Doctoral Dissertation, Mechanical Engineering Department, Columbia University, New York, New York, U.S.A.

ANALYSIS

For planar solidification of a saturated liquid, the temperature distribution in the solidified material, T, satisfies the transient heat conduction equation in time t and space x as

$$\alpha \frac{\partial^2 T}{\partial X^2} = \frac{\partial T}{\partial t} \tag{1}$$

where the thermal diffusivity of the solidified material, α, is introduced and constant properties are assumed. The temperature distribution equals the freezing temperature, T_f, at the freezing front, $X = X_f$:

$$T(X = X_f, t) = T_f. \tag{2}$$

The boundary condition at the wall is given in terms of the wall conductance, H_w, and the characteristic value of the temperature in the flowing coolant, T_c, yielding

$$H_w[T(X = 0, t) - T_c] = k \left.\frac{\partial T}{\partial X}\right|_{X=0} \tag{3}$$

where the thermal conductivity of the solidified material, k, is introduced. The last boundary condition to be considered is the energy balance at the freezing front, which yields in terms of the latent heat of fusion, L, and the density of the solidified material, ρ:

$$\rho L \frac{dX_f}{dt} = k \left.\frac{\partial T}{\partial X}\right|_{X=X_f}. \tag{4}$$

At this point, it is convenient to introduce the dimensionless quantities: position, x, freezing-front position, x_f, time, t, temperature, u, and the physical parameter, ε; all quantities defined as

$$x = \frac{H_w X}{k}, \qquad x_f = \frac{H_w X_f}{k}, \qquad \tau = \frac{H_w^2(T_f - T_c)t}{\rho L k}$$

$$u = \frac{T - T_c}{T_f - T_c}, \qquad \varepsilon = \frac{c(T_f - T_c)}{L} \tag{5}$$

where the specific heat of the solidified material, c, is introduced. Substitution of equations (5) into equations (1)–(4) and a change of variables from x, τ to x, x_f yields the normalized form of the boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2} = \varepsilon \left.\frac{\partial u}{\partial x}\right|_{x=x_f} \frac{\partial u}{\partial x_f}$$

$$u(x = x_f, x_f) = 1, \qquad u(x = 0, x_f) = \left.\frac{\partial u}{\partial x}\right|_{x=0} \tag{6}$$

where the position of the freezing front is calculated from

$$\frac{\mathrm{d}x_f}{\mathrm{d}\tau} = \left.\frac{\partial u}{\partial x}\right|_{x=x_f}. \tag{7}$$

A perturbation solution is now assumed for the temperature distribution in terms of the parameter ε:

$$u(x, x_f; \varepsilon) = u_i(x, x_f)\varepsilon^{i-1} \tag{8}$$

where

$$i = 1, 2, \ldots, N_t$$

where the summation convention over repeated indices is adopted in equation (8) and in the remainder of this work, unless it is otherwise noted. The number of terms desired in the solution is indicated as N_t. Substituting equation (8) into equations (6) and equating coefficients of equal powers of ε, one obtains

$$\frac{\partial^2 u_i}{\partial x^2} = \begin{cases} 0 \text{ for } i = 1 \\ \left.\dfrac{\partial u_j}{\partial x}\right|_{x=x_f} \dfrac{\partial u_{i-j}}{\partial x_f}, \quad j = 1, 2, \ldots, i-1 \text{ for } i \geq 2 \end{cases}$$

$$u_i(x = x_f, x_f) = \begin{cases} 1 \text{ for } i = 1 \\ 0 \text{ for } i \geq 2 \end{cases}, \qquad u_i(x = 0, x) = \left.\frac{\partial u_i}{\partial x}\right|_{x=0} \tag{9}$$

where

$$i = 1, 2, \ldots, N_t.$$

The solution to the differential equations in (9) can easily be obtained, yielding

$$u_i = \lambda_{i,j,1} x^{j-1} \tag{10}$$

where

$$i = 1, 2, \ldots, N_t, \qquad j = 1, 2, \ldots, 2i$$

and the coefficients $\lambda_{i,j,1}$ are functions of x_f alone, which are calculated from the boundary conditions and sequence relations in equations (9).

A presentation in the form of a summary follows for the solution to all the coefficients $\lambda_{i,j,1}$; however, first define the terms $a_{i,j}$ as the binomial coefficients:

$$a_{i,1} = a_{i,i} = 1 \text{ for } i = 2, 3, \ldots \text{ no sum on } i$$

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j} \text{ for } i = 3, 4 \ldots, \qquad j = 2, 3, \ldots, i-1. \tag{11}$$

Also for convenience define the function $f_{j,1} = x_f^{j-1}$ and note that its $(m-1)$th derivative is given by

$$f_{j,m} = \begin{cases} (j-1)(j-2)\ldots(j-m+1)x_f^{j-m} \text{ for } m = 2, 3, \ldots, j \\ 0 \text{ for } m \geq j+1 \qquad \text{no sum on } j \text{ or } m. \end{cases} \tag{12}$$

In the manner described schematically, for the first few terms, in Fig. 1, the following set of equations is used to

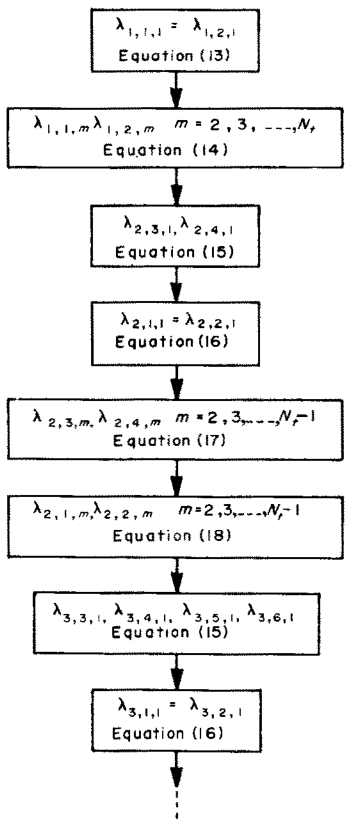

FIG. 1. Procedure for calculation of coefficients, $\lambda_{i,j,1}$, for perturbation solution, equation (10), for any given value of normalized freezing front position, x_f.

calculate as many coefficients $\lambda_{i,j,1}$ as desired (i.e. $i = 1, 2, \ldots, N_t$ and $j = 1, 2, \ldots, 2i$):

$$\lambda_{1,1,1} = \lambda_{1,2,1} = \frac{1}{1 + x_f} \tag{13}$$

$$\lambda_{1,1,m} = \lambda_{1,2,m} = (-1)^{m-1} \frac{(m-1)(m-2)\ldots(1)}{(1 + x_f)^m} \tag{14}$$

where
$m = 2, 3, \ldots, N_t$, no sum on m

$$\lambda_{1,l+2,1} = (k-1)\lambda_{j,k,1} x_f^{k-2} \frac{\lambda_{i-j,l,2}}{l(l+1)} \tag{15}$$

where
$i = 2, 3, \ldots, N_t$, $l = 1, 2, \ldots, 2(i-1)$

$j = 1, 2, \ldots, i - \frac{1}{2}\left[l + \frac{1 + (-1)^{l+1}}{2}\right]$, $k = 2, 3, \ldots, 2i$

no sum on l

$$\lambda_{i,1,1} = \lambda_{i,2,1} = \frac{\lambda_{i,j,1} x_f^{j-1}}{1 + x_f} \tag{16}$$

where

$i = 2, 3, \ldots, N_t$, $j = 3, 4, \ldots 2i$

$$\lambda_{i,l+2,m} = a_{m,m-n+1} a_{n,n-h+1} \lambda_{j,k,h} f_{k,n-h+2} \frac{\lambda_{i-j,l,m-n+2}}{l(l+1)} \tag{17}$$

where

$i = 2, 3, \ldots, N_t - 1$, $l = 1, 2, \ldots, 2(i-1)$

$j = 1, 2, \ldots, i - \frac{1}{2}\left[l + \frac{1 + (-1)^{l+1}}{2}\right]$, $k = 2, 3, \ldots, 2i$

$m = 2, 3, \ldots, N_t - i + 1$, $n = 1, 2, \ldots, m$

$h = 1, 2, \ldots, n$, no sum on m

$$\lambda_{i,1,m} = \lambda_{i,2,m} = -a_{m,m-n+1} a_{n,n-h+1} \lambda_{i,j,h} f_{j,n-h+1} \lambda_{1,1,m-n+1} \tag{18}$$

where

$i = 2, 3, \ldots, N_t - 1$, $j = 3, 4, \ldots, 2i$

$m = 2, 3, \ldots, N_t - i + 1$, $n = 1, 2, \ldots, m$

$h = 1, 2, \ldots, n$, no sum on m.

The calculations for $\lambda_{i,j,1}$ (for a given value of x_f) begin with equations (13) and (14), then equations (15)–(18) are used repeatedly in the order as shown, for the first few terms. in Fig. 1. Equations (11) and (12) have not been included as part of this procedure since their values can first be calculated and stored for future use.

Equations (7), (8) and (10) combine to yield for the derivative of time with respect to the freezing-front position:

$$\frac{d\tau}{dx_f} = \frac{d\tau_i}{dx_f}\varepsilon^{i-1}, \qquad i = 1, 2, \ldots, N_t$$

$$\frac{d\tau_i}{dx_f} = \begin{cases} \frac{1}{g_1} \text{ for } i = 1 \\ -\frac{g_2}{g_1^2} \text{ for } i = 2 \\ -\frac{1}{g_1^2}\left(g_i + g_{j+1}\frac{d\tau_{i-j}}{dx_f}\right) \text{ for } i = 3, 4, \ldots N_t \\ \text{where } j = 1, 2, \ldots i - 2 \end{cases} \tag{19}$$

$g_i = (j-1)x_f^{j-2}\lambda_{i,j,1}$
where $i = 1, 2, \ldots N_t$, $j = 2, 3, \ldots, 2i$.

The number of mathematical operations increases rapidly as more terms, N_t, are included in the calculation. However, these operations can easily be done by a digital computer; thereby, obtaining very accurate results. Note that the procedure up to this point is exact except for numerical truncation errors by the digital computer. No numerical methods or any other approximations have been made. The only requirement is that the series solution, equation (10), converges or somehow that its summation is possible.

For a given value of x_f, equations (19) can be evaluated once the values of the functions $\lambda_{i,j,1}$ are known. Numerical integration of the terms $d\tau_i/dx_f$ in equations (19) yields the coefficients τ_i in the perturbation series for the freezing-time solution. It is convenient now to define the sequence of partial sums:

$$\tau_i' = \tau_j \varepsilon^{j-1} \tag{20}$$

where

$i = 1, 2, \ldots, N_t$, $\quad j = 1, 2, \ldots, i$.

RESULTS AND DISCUSSION

Table 1 shows the first nine terms, τ_i, $i = 1, 2, \ldots, 9$, of the series for the normalized freezing time, τ, for values of the normalized front position up to $x_f = 5$. The numerical integration of the terms $d\tau_i/dx_f$ was performed with Simpson's rule with increments in front position $\Delta x_f = 0{\cdot}02$. Successive-order terms are seen to alternate in sign. For

Table 1. First nine terms, $\tau_i(x_f)$, $i = 1, 2, \ldots, 9$, in the regular-perturbation solution for the freezing time for planar solidification with convection at the wall and liquid at the freezing temperature

x_f \ i	1	2	3	4	5	6	7	8	9
0·2	0·2200	0·01778	−0·001564	0·0003945	−0·0001383	0·00005842	−0·00002793	0·00001462	−0·000008213
0·4	0·4800	0·06476	−0·008154	0·002846	−0·001335	0·0007332	−0·0004438	0·0002869	−0·0001940
0·6	0·7800	0·1350	−0·01932	0·007488	−0·003804	0·002211	−0·001389	0·0009138	−0·0006180
0·8	1·120	0·2252	−0·03398	0·01368	−0·007083	0·004134	−0·002573	0·001661	−0·001096
1	1·500	0·3333	−0·05139	0·02092	−0·01083	0·006253	−0·003828	0·002422	−0·001576
1.4	2·380	0·5989	−0·09282	0·03768	−0·01918	0·01082	−0·006461	0·004001	−0·002643
1·8	3.420	0·9257	−0·1418	0·05692	−0·02852	0·01585	−0·009353	0·005750	−0·004005
2·2	4·620	1·311	−0·1979	0·07854	−0·03893	0·02145	−0·01259	0·007719	−0·005770
2·6	5·980	1·753	−0·2608	0·1025	−0·05048	0·02767	−0·01620	0·009923	−0·007997
3·0	7·500	2·250	−0·3305	0·1292	−0·06323	0·03456	−0·02021	0·01237	−0·01071
4	12·00	3·733	−0·5347	0·2068	−0·1006	0·05479	−0·03199	0·01956	−0·01963
5	17·50	5·555	−0·7823	0·3008	−0·1459	0·07939	−0·04632	0·02832	−0·03153

terms of the same sign, the higher the order the lower its magnitude. Therefore, the series solution including nine terms is convergent for values of the perturbation parameter $\varepsilon \leqslant 1$. For example if $x_f = 1$ and $\varepsilon = 0·5$, then from Table 1 and equation (20) it can be calculated that $\tau_1^t = 1·500$, $\tau_2^t = 1·667$, $\tau_3^t = 1·654$ and $\tau_i^t = 1·656$ for $i \geqslant 4$. The corresponding solution from Goodman's [2] heat balance integral technique is $\tau = 1·654$. As another example, if $x_f = 1$ and $\varepsilon = 1$, then the freezing-time solutions are as follows:

i	1	2	3	4
τ_i^t	1·500	1·833	1·782	1·811

5	6	7	8	9
1·800	1·807	1·803	1·805	1·804

The solution from Goodman in this case is $\tau = 1·789$.

It is simple to calculate algebraic expressions for the first three terms of the perturbation solution considered above. Shanks [3] transformations are found to increase considerably the rate and range of convergence of the perturbation solution. The solution after application of the first non-linear transformation of Shanks can easily be expressed in algebraic form. For further information the reader should consult with [4] where in addition a comparison is found with the perturbation technique of Lock [5].

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